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Strong convergence of the new modified composite iterative method for strict pseudo-contractions in Hilbert spaces

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Abstract. In this paper, we introduce and analyze a new modified Mann iterative scheme for strict pseudo-contraction mappings in Hilbert spaces. The results presented in this paper improve and extend the main results in [1] and many others.

Keywords: Strong convergence, Strict pseudo-contractions, Composite iterative scheme, Hilbert space

MSC 2000 classification: 47H07, 47H09, 47H10, 47J25

1 Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. Let I be the identity mapping on H and C a closed convex subset of H .

A mapping T of H into itself is called a k -strict pseudo-contraction mapping, if $\forall x, y \in K$, $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2$, here $0 \leq k < 1$. We use $F(T)$ to denote the set of fixed points of T (i.e. $F(T) = \{x \in K : Tx = x\}$).

In Hilbert spaces, it is clear that a k -strict pseudo-contraction mappings is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - k}{2} \|(I - T)x - (I - T)y\|^2, \quad (1)$$

i.e.

$$\frac{1 - k}{2} \|(I - T)x - (I - T)y\|^2 \leq \langle (I - T)x - (I - T)y, x - y \rangle. \quad (2)$$

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Remark. Notice that a mapping $T : H \rightarrow H$ is called *non-expansive* mappings, if for all $x, y \in H$, $\|Tx - Ty\| \leq \|x - y\|$. Therefore, a non-expansive mapping T is a 0-strict pseudo-contractive mapping.

A linear bounded operator B is strongly positive if there exists a constant $\bar{\gamma} > 0$ with property $\langle Bx, x \rangle \geq \bar{\gamma} \|x\|^2$, $\forall x \in H$.

Marino and Xu [2] introduced a new iterative scheme by the viscosity approximation method:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \quad (3)$$

where, $S : H \rightarrow H$ is a non-expansive mapping. They proved that the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{p \in F(S)} \frac{1}{2} \langle Bp, p \rangle - h(p), \quad \forall p \in F(S),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

The normal Mann's iterative process was introduced by Mann [3] in 1953 as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \quad (4)$$

where $\{\alpha_n\}$ is a real number sequence in $(0,1)$.

If T is a non-expansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the normal Mann's iterative process (1.4) weakly converges to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [4], or more generally, in a uniformly convex Banach space such that its dual has the KK property as proved by Garcia Falset, Kaczor, Kuczumow and Reich in [5]). However, this scheme has only weak convergence even in a Hilbert space [6]. Therefore, many authors try to modify normal Mann's iteration process to have strong convergence; see, e.g., [7-12, 13, 14] and the references therein.

Yao et al. [14] considered the following iteration process.

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \end{cases} \quad n \geq 0, \quad (5)$$

where T is a non-expansive mapping of C into itself and f is an α -contraction (i.e. $\|f(x) - f(y)\| \leq \alpha \|x - y\|$, $0 \leq \alpha < 1$). They proved the sequence $\{x_n\}$ defined by (5) strongly converges to a fixed point of T provided the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Motivated by Marino and Xu[2,9] and Yao et al. [14], Marino et al. [1] introduced a composite iteration scheme as follows:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) \sum_{i=1}^N \eta_i T_i x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n, \quad n \geq 0, \end{cases} \quad (6)$$

where f is an α -contraction, γ is a suitable coefficient and B is a linear bounded strongly positive operator, T_i is a k_i -pseudo-contraction with $0 \leq k_i < 1$ and η_i is a positive constant such that $\eta_1 + \eta_2 + \dots + \eta_N = 1$. They proved, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ defined by (6) converges to a common fixed point of $\{T_1, T_2, \dots, T_N\}$, which solves some variation inequality. To be more precisely, they obtained the next Theorems.

Theorem M1. [1]. Let H be a Hilbert space and let for $i = 1, 2, \dots, N$, T_i be a k_i -strict pseudo-contraction on H for some $0 \leq k_i < 1$ with $\Omega = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and f be an α -contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in $(0, 1)$, satisfying the following conditions

$$(M1) \quad \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^\infty \alpha_n = \infty.$$

$$(M2) \quad \sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty.$$

$$(M3) \quad 0 \leq \max_i \{k_i\} \leq \beta_n \leq \beta < 1 \text{ for all } n \geq 0.$$

Then $\{x_n\}$ defined by (6) converges strongly to some common fixed point q of $\{T_1, T_2, \dots, T_N\}$, which solves the following variational inequality:

$$\langle \gamma f q - B q, p - q \rangle \leq 0, \quad \forall p \in \Omega.$$

Theorem M2. [1]. Let H be a Hilbert space and let for $i = 1, 2, \dots, N$, T_i be a k_i -strict pseudo-contraction on H for some $0 \leq k_i < 1$ with $\Omega = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and f be an α -contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^\infty$, $\{\eta_i^{(n)}\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in $(0, 1)$, satisfying the following conditions

$$(M1') \quad \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^\infty \alpha_n = \infty;$$

$$(M2') \quad \sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty;$$

(M3') for every fixed n , $\sum_{i=1}^N \eta_i^{(n)} = 1$ and $\inf_n \eta_i^{(n)} > 0$;

(M4') $0 \leq \max_i \{k_i\} \leq \beta_n \leq \beta < 1$ for all $n \geq 0$;

(M5') $\sum_{n=0}^{\infty} |\eta_i^{(n+1)} - \eta_i^{(n)}| < \infty$ (for $i = 1, 2, \dots, N$).

Let $\{x_n\}$ be defined by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) \sum_{i=1}^N \eta_i^{(n)} T_i x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n, n \geq 0, \end{cases} \quad (7)$$

then $\{x_n\}$ converges strongly to the common fixed point q of $\{T_1, T_2, \dots, T_N\}$, which solves the following variational inequality:

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \Omega.$$

Inspired by Marino et al. [1], in this paper, our purpose is to introduce a modified composite iterative algorithm (given in next section 3) to approximate a common fixed point of finite family of strict pseudo-contraction mappings, which solves some variational inequality. Our results improve and extend the results of Marino et al. [1], Kim and Xu [8], Marino and Xu [2], Yao et al. [14].

2 Preliminaries

Lemma 1. ([15]). Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following condition:

$$a_{n+1} \leq (1 - \lambda_n) a_n + b_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer and $\{\lambda_n\} \subset (0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2. ([16]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n) x_n$ for all integers $n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 3. ([17]). Let E be a real Banach space and $J : E \rightarrow 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$ the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).$$

Especially, when $E = H$, then $J = I$, so from Lemma 3 we have that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Lemma 4. ([2]). Assume that A is a strong positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 5. (Marino and Xu [2]). Let H be a real Hilbert space. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Let f be an α -contraction. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $T : H \rightarrow H$ be a non-expansive mapping. For $t \leq \|A\|^{-1}$, let x_t be the fixed point of the contraction $x_t \rightarrow \gamma f(x) + (I - tA)Tx$. Then $\{x_t\}$ converges strongly as $t \rightarrow 0$ to a fixed point \bar{x} of T , which solves the variational inequality

$$\langle \gamma f\bar{x} - B\bar{x}, z - \bar{x} \rangle \leq 0, \quad \forall z \in F(T).$$

Lemma 6. (Acedo and Xu [7]). Let H be a real Hilbert space, K a closed convex subset of H . Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $T_i : K \rightarrow K$ is a k_i -strict pseudo-contraction for some $0 \leq k_i < 1$. Assume $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $\sum_{i=1}^N \eta_i T_i$ is a k -strict pseudo-contraction, with $k = \max\{k_i : 1 \leq i \leq N\}$.

Lemma 7. (Acedo and Xu [7]). Let $\{T_i\}$ and $\{\eta_i\}$ be given as in Lemma 6. Suppose that $\{T_i\}$ has a common fixed point. Then $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$.

Lemma 8. (Zhou [18]). Let $T : H \rightarrow H$ be a k -strict pseudo-contraction. Define $S : H \rightarrow H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for each $x \in H$. Then, as $\lambda \in [k, 1)$, S is non-expansive such that $F(S) = F(T)$.

A Banach space E is said to satisfy Opial's condition [19] if, for any $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, the following inequality holds:

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|,$$

for all $y \in E$ with $y \neq x$. It is well-known that Hilbert spaces satisfies Opial's condition.

3 Main results

Theorem 1. Let H be a real Hilbert space and let T be a k -strict pseudo-contraction on H for some $0 \leq k < 1$ with $F(T) \neq \emptyset$ and f be an α -contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1)$, satisfying the following conditions

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Let $\{x_n\}$ be defined by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n, \quad n \geq 0, \end{cases} \quad (8)$$

where $S = \sigma I + (1 - \sigma)T$, $k \leq \sigma < 1$, then $\{x_n\}$ defined by Theorem 1 converges strongly to a fixed point q of T , which solves the following variational inequality:

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in F(T).$$

PROOF.

Step 1. Prove $\{x_n\}$ is bounded. Since $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, without loss of generality, we may assume that $\alpha_n < \|B\|^{-1}$ for all $n \geq 0$. From Lemma 4, we know that $\|I - \alpha_n B\| \leq 1 - \alpha_n \bar{\gamma}$. For all $p \in F(T)$, since S is a non-expansive mapping and $F(S) = F(T)$ by Lemma 8, we have

$$\|y_n - p\|^2 = \|\beta_n(x_n - p) + (1 - \beta_n)(Sx_n - p)\|^2 \leq \|x_n - p\|^2,$$

and

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Bp) + (I - \alpha_n B)(y_n - p)\| \\ &\leq \alpha_n \|\gamma f(x_n) - Bp\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\ &\leq \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\ &\leq (1 - \alpha_n(\bar{\gamma} - \gamma \alpha)) \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| \\ &\leq \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - Bp\|}{\bar{\gamma} - \gamma \alpha}\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded, so is $\{y_n\}$.

Step 2. Prove $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Set $\gamma_n = 1 - \beta_n$ and $v_n = \frac{x_{n+1} - x_n + \gamma_n x_n}{\gamma_n} = \frac{\alpha_n(\gamma f(x_n) - B y_n)}{1 - \beta_n} + S x_n$, then

$$\begin{aligned} \|v_{n+1} - v_n\| &\leq \frac{\alpha_n}{1 - \beta_n} M + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} M + \|S x_{n+1} - S x_n\| \\ &\leq \frac{\alpha_n}{1 - \beta_n} M + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} M + \|x_{n+1} - x_n\| \end{aligned} \quad (9)$$

where M is a constant satisfying $\|\gamma f(x_n) - B y_n\| \leq M$ for $n \geq 1$. It follows from (9), $\limsup_{n \rightarrow \infty} \{\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\} = 0$, which implies that $\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0$ by Lemma 2. From the definition of v_n we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (10)$$

Step 3. Prove $\|Tx_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. In fact, from (8) and (10) we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0, \quad \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

Hence

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = \lim_{n \rightarrow \infty} \frac{1}{1 - \beta_n} \|x_n - y_n\| = 0,$$

which implies that

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = \lim_{n \rightarrow \infty} (1 - \sigma) \|Tx_n - x_n\| = 0. \quad (11)$$

Step 4. Let $q = \lim_{t \rightarrow 0^+} x_t$, where x_t is the fixed point (for $t \in (0, \|B\|^{-1})$) of contraction $x \mapsto tx + (I - tB)Sx$. From Lemma 5 and Lemma 8, $q \in F(S) = F(T)$, and

$$\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad \forall p \in F(T) \quad (12)$$

Claim $\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Bq, x_n - q \rangle \leq 0$. Take the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Bq, x_n - q \rangle = \lim_{j \rightarrow \infty} \langle \gamma f(q) - Bq, x_{n_j} - q \rangle. \quad (13)$$

Since $\{x_{n_j}\}$ is bounded, so there exists a subsequence of $\{x_{n_j}\}$ such that it converges weakly to a point $p \in K$. Without loss generality, let $\{x_{n_j}\}$ denote it and $x_{n_j} \rightharpoonup p$. By (11) and Opial's condition, $p \in F(S) = F(T)$. Thus from (12) and (13) we have that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Bq, x_n - q \rangle \leq 0. \quad (14)$$

Step 5. Prove that $\{x_n\}$ converges strongly to q . It follows from Lemma 3 and (8) that

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n(\gamma f(x_n) - Bq) + (I - \alpha_n B)(y_n - q)\|^2 \\ &\leq (1 - \bar{\gamma}\alpha_n)^2 \|y_n - q\|^2 + 2\alpha_n \langle \gamma f(x_n) - Bp, x_{n+1} - q \rangle \\ &= (1 - \bar{\gamma}\alpha_n)^2 \|x_n - q\|^2 \\ &\quad + 2\alpha_n \langle \gamma f(x_n) - \gamma f(q) + \gamma f(q) - Bq, x_{n+1} - q \rangle \\ &\leq (1 - \bar{\gamma}\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \alpha \gamma \|x_n - q\| \|x_{n+1} - q\| \\ &\quad + 2\alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\ &\leq (1 - \bar{\gamma}\alpha_n)^2 \|x_n - q\|^2 + \alpha_n \alpha \gamma (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \\ &\quad + 2\alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle, \end{aligned}$$

which yields that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1 - \alpha_n \frac{2\bar{\gamma} - 2\alpha\gamma}{1 - \alpha_n\alpha\gamma}) \|x_n - q\|^2 \\ &+ \frac{\alpha_n^2 \bar{\gamma}^2}{1 - \alpha_n\alpha\gamma} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n\alpha\gamma} \langle \gamma f(q) - Bq, x_{n+1} - q \rangle. \end{aligned} \quad (15)$$

By boundness of $\{x_n\}$ and the condition (i) and Lemma 1, $\{x_n\}$ converges strongly to q . This completes the proof of Theorem 1.

□ QED

Next, we give strong convergence theorems for a finite family of strict pseudo-contractions.

Theorem 2. *Let H be a real Hilbert space and for $i = 1, 2, \dots, N$, let T_i be a k_i -strict pseudo-contraction on H for some $0 \leq k_i < 1$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and f be an α -contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$, $\eta_i \in (0, 1)$ and $\sum_{i=1}^N \eta_i = 1$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in $(0, 1)$, satisfying the following conditions*

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Let $\{x_n\}$ be defined by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n, \end{cases} \quad n \geq 0, \quad (16)$$

where $S = \sigma I + (1 - \sigma) \sum_{i=1}^N \eta_i T_i$, $k = \max\{k_i : 1 \leq i \leq N\} \leq \sigma < 1$, then $\{x_n\}$ converges strongly to a common fixed point q of $\{T_1, T_2, \dots, T_N\}$, which solves the following variational inequality:

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

PROOF. Since $\sum_{i=1}^N \eta_i T_i$ is a k -strict pseudo-contraction mapping, by Theorem 1 we know Theorem 2 is true. This completes the proof of Theorem 2.

□ QED

Theorem 3. *Let H be a real Hilbert space and for $i = 1, 2, \dots, N$, let T_i be a k_i -strict pseudo-contraction on H for some $0 \leq k_i < 1$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and f be an α -contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^\infty$, $\{\eta_i^{(n)}\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in $(0, 1)$, satisfying the following conditions*

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty;$
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1;$
- (iii) $\sum_{i=1}^N \eta_i^{(n)} = 1, \inf_n \eta_i^{(n)} > 0, \lim_{n \rightarrow \infty} |\eta_i^{(n+1)} - \eta_i^{(n)}| = 0$ for $i = 1, 2, \dots, N$.

Let $\{x_n\}$ be defined by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n, \end{cases} \quad n \geq 0, \quad (17)$$

where $S_n = \sigma I + (1 - \sigma) \sum_{i=1}^N \eta_i^{(n)} T_i$, $k = \max\{k_i : 1 \leq i \leq N\} \leq \sigma < 1$, then $\{x_n\}$ converges strongly to a common fixed point q of $\{T_1, T_2, \dots, T_N\}$, which solves the following variational inequality:

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \cap_{i=1}^N F(T_i).$$

PROOF. For each n , let $H_n = \sum_{i=1}^N \eta_i^{(n)} T_i$, then H_n is a k -strict pseudo-contraction by Lemma 6, $k = \max\{k_i : 1 \leq i \leq N\}$. Moreover, $S_n = \sigma I + (1 - \sigma) H_n$ is a non-expansive mapping and $F(S_n) = F(H_n) = \cap_{i=1}^N F(T_i)$ by Lemma 8.

Step 1. Let $p \in \cap_{i=1}^N F(T_i)$, from (17), we have

$$\|y_n - p\|^2 = \|\beta_n(x_n - p) + (1 - \beta_n)(S_n x_n - p)\|^2 \leq \|x_n - p\|^2,$$

and

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Bp) + (I - \alpha_n B)(y_n - p)\| \\ &\leq \alpha_n \|\gamma f(x_n) - Bp\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\ &\leq \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\ &\leq (1 - \alpha_n(\bar{\gamma} - \gamma \alpha)) \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| \\ &\leq \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - Bp\|}{\bar{\gamma} - \gamma \alpha}\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded, so is $\{y_n\}$.

Step 2. Prove $\|x_{n+1} - x_n\| \rightarrow 0$. Let

$$\gamma_n = 1 - \beta_n, v_n = \frac{x_{n+1} - x_n + \gamma_n x_n}{\gamma_n} = \frac{\alpha_n(\gamma f(x_n) - B y_n)}{\gamma_n} + S_n x_n.$$

Let M_3 be a constant such that

$$\{\|\gamma f(x_n) - B y_n\|, \|T_1 x_n\|, \|T_2 x_n\|, \dots, \|T_N x_n\|\} \leq M_3$$

for $n \geq 0$. By $S_n = \sigma I + (1 - \sigma) \sum_{i=1}^N \eta_i^{(n)} T_i$, we have

$$S_{n+1}x_n - S_nx_n = (1 - \sigma) \sum_{i=1}^N (\eta_i^{(n+1)} - \eta_i^{(n)}) T_i x_n.$$

Then

$$\|S_{n+1}x_{n+1} - S_nx_n\| \leq \|x_{n+1} - x_n\| + M_3 \sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}|.$$

Furthermore,

$$\|v_{n+1} - v_n\| \leq \frac{\alpha_n M_3}{\gamma_n} + \frac{\alpha_{n+1} M_3}{\gamma_{n+1}} + \|x_{n+1} - x_n\| + M_3 \sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}|,$$

which implies that $\limsup_{n \rightarrow \infty} \{\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\} = 0$. Using Lemma 2, we obtain $\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0$, this shows that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (18)$$

Again from (17) and (18), we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0, \quad \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0.$$

Then

$$\lim_{n \rightarrow \infty} \|S_n x_n - x_n\| = \lim_{n \rightarrow \infty} \frac{1}{1 - \beta_n} \|y_n - x_n\| = 0. \quad (19)$$

Step 3. Prove

$$\limsup_{n \rightarrow \infty} \langle \gamma f q - Bq, x_n - q \rangle \leq 0, \quad q \in \cap_{i=1}^N F(T_i),$$

where q is the unique solution of the following variational inequality

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \cap_{i=1}^N F(T_i). \quad (20)$$

Take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle \gamma f q - Bq, x_n - q \rangle = \lim_{n \rightarrow \infty} \langle \gamma f q - Bq, x_{n_j} - q \rangle.$$

For each i , since $\eta_i^{(n)}$ is bounded, there exists a subsequence of $\eta_i^{(n_j)}$ is still denoted by $\eta_i^{(n_j)}$ such that $\eta_i^{(n_j)} \rightarrow \eta_i \in (0, 1]$ as $j \rightarrow \infty$. At the

same time, since $\{x_{n_j}\}$ is bounded, there exists a subsequence of $\{x_{n_j}\}$ is still denoted by $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup z$ as $j \rightarrow \infty$. Define mapping $S = \sigma I + (1 - \sigma) \sum_{i=1}^N \eta_i T_i$, then S is also non-expansive mapping and $F(S) = F(S_n) = \cap_{i=1}^N F(T_i)$ by Lemma 7 and 8. Moreover, for all $x \in C$,

$$\|S_{n_j}x - Sx\| \leq \sum_{i=1}^N |\eta_i^{(n_j)} - \eta_i| \|T_i x\| \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (21)$$

We claim $z \in F(S)$. If not so, i.e. $z \neq Sz$. Then using the Opial's condition, from (19) and (21) we obtain

$$\begin{aligned} \liminf_{j \rightarrow \infty} \|x_{n_j} - z\| &< \liminf_{j \rightarrow \infty} \|x_{n_j} - Sz\| \\ &\leq \liminf_{j \rightarrow \infty} (\|x_{n_j} - S_{n_j}x_{n_j}\| + \|S_{n_j}x_{n_j} - S_{n_j}z\| + \|S_{n_j}z - Sz\|) \\ &\leq \liminf_{j \rightarrow \infty} \|x_{n_j} - z\|. \end{aligned} \quad (22)$$

This is a contradiction. So $z \in F(S) = \cap_{i=1}^N F(T_i)$. It follows from (20) that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \gamma f q - Bq, x_n - q \rangle \\ = \lim_{j \rightarrow \infty} \langle \gamma f q - Bq, x_{n_j} - q \rangle = \langle \gamma f q - Bq, z - q \rangle \leq 0. \end{aligned} \quad (23)$$

Step 4. Prove $\{x_n\}$ converge strongly to q , where q is the unique solution of the following variational inequality

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \cap_{i=1}^N F(T_i).$$

Reasoning as in Step 5 of Theorem 1, we obtain $\{x_n\}$ converges strongly to q . This completes the proof of Theorem 2. \square

Remark 1. Since a non-expansive mapping is a 0-strict pseudo-contraction mapping, our results are suitable to non-expansive mappings. In addition, our results remove the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ imposed on parameter $\{\alpha_n\}$ in [1,2,7,8]. We remove also the condition $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ imposed on parameter $\{\beta_n\}$ in [1].

Remark 2. The advantages of these results in the present paper are that fewer restrictions are imposed on the parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\eta_i^{(n)}\}$. All of the results obtained in this paper can be viewed as a supplement to the results obtained in [1,2,7,8,14]

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